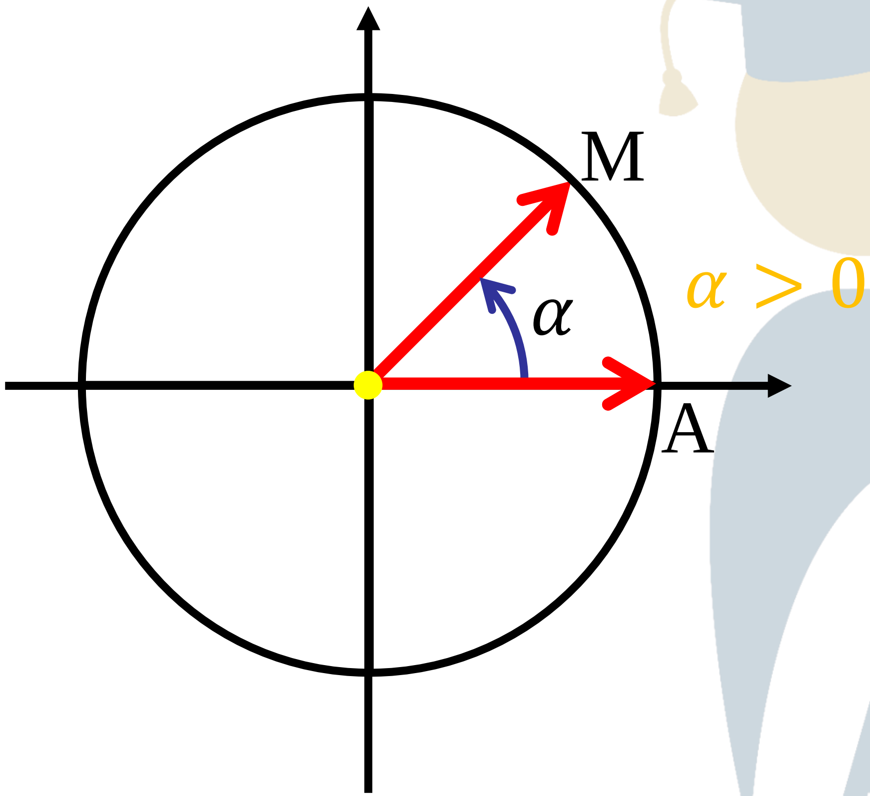
Elementary trigonometric equations

Recall (directed arcs)

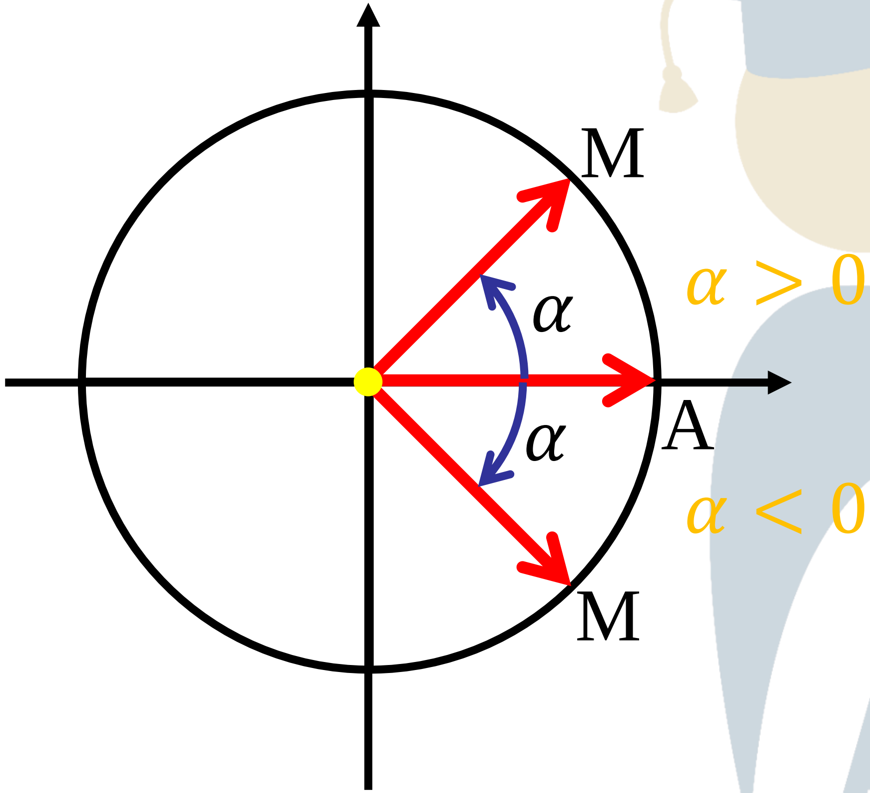


$\overset{\frown}{AM}$ is an arc of measure α.

$\overset{\frown}{AM} = \alpha + 2k\pi \quad ; \quad k \in \mathbb{Z}$

$\alpha \in \left]-\pi ; \pi\right]$

Recall (directed arcs)



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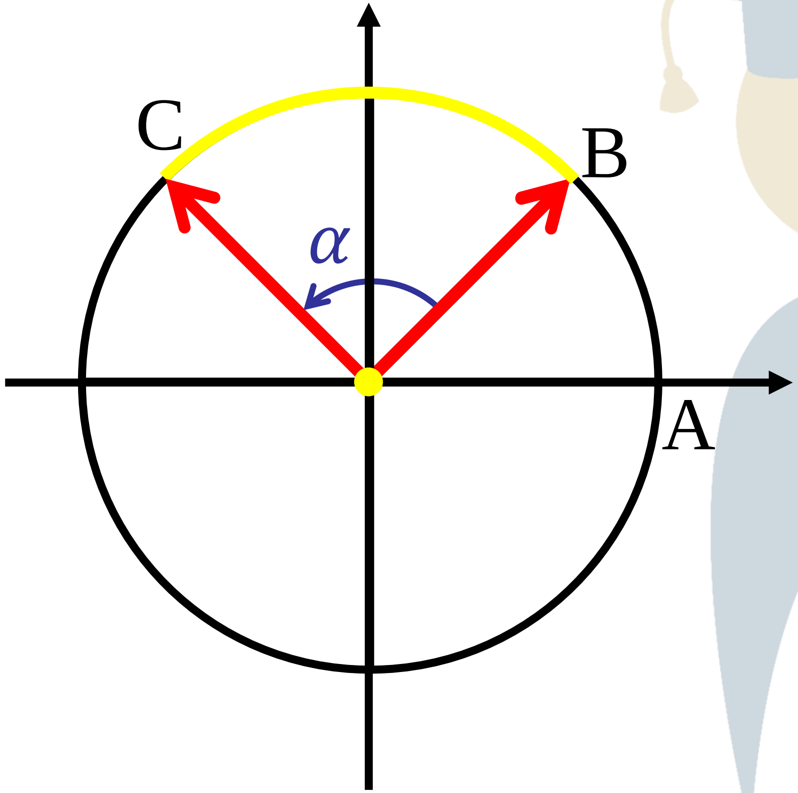
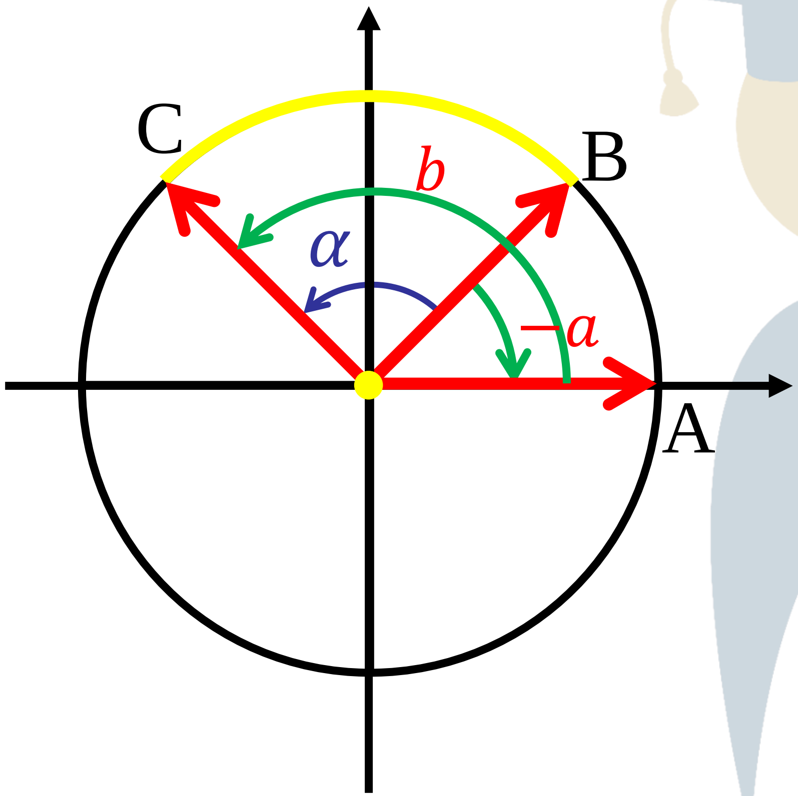
Chasles relation for directed arc



The measure of the directed arc $\widehat{BC}$ is obtained by moving from B to C directly.

Chasles relation for directed arc



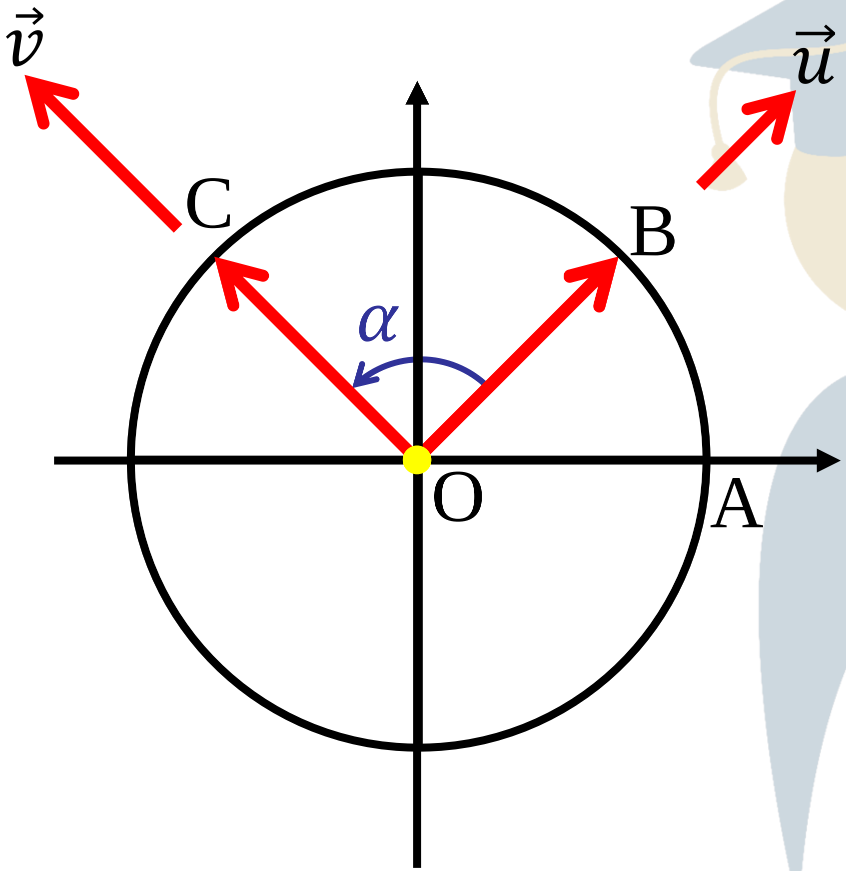
The measure of the directed arc $\overset{\frown}{BC}$ is obtained by moving from B to C directly.

$$\overset{\frown}{BC} = \overset{\frown}{BA} + \overset{\frown}{AC}$$

$$\alpha \qquad -a \qquad +b$$

$$\alpha = b - a + 2k\pi \ ; \quad k \in \mathbb{Z}$$

Angle between two vectors



$\vec{u}$ and $\vec{v}$ are two non zero vectors.

The directed angle between them is denoted by:

$$(\vec{u};\vec{v})$$

$$(\vec{u};\vec{v}) = \left(\overrightarrow{OB};\overrightarrow{OC}\right) = mes\,\widehat{BC}$$

Remark:

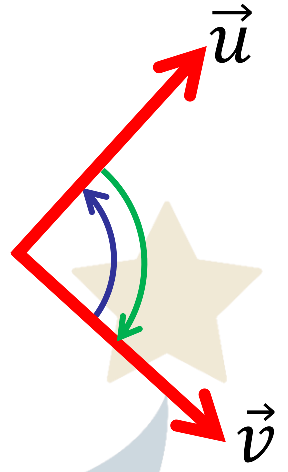
According to Chasles rule:

$$(\vec{u};\vec{w}) + (\vec{w};\vec{v}) = (\vec{u};\vec{v})$$

Properties

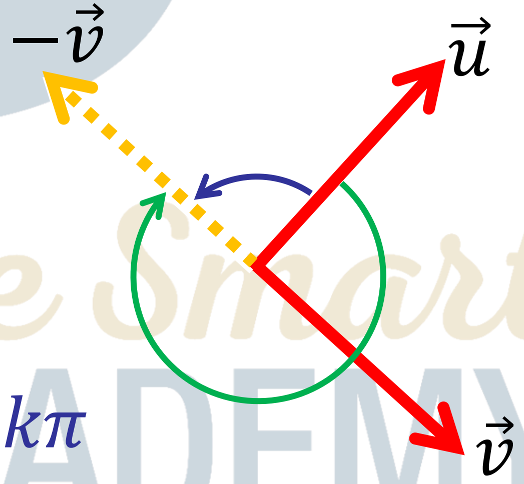
❶ $(\vec{u};\vec{v}) = -(\vec{v};\vec{u}) + 2k\pi$

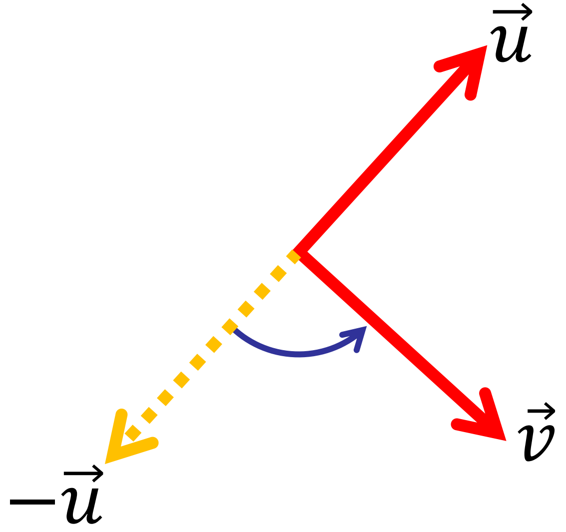
$$(\vec{u};\vec{v}) + (\vec{v};\vec{u}) = (\vec{u};\vec{u})$$
$$= 0 + 2k\pi = 2k\pi$$



❷ $(\vec{u};-\vec{v}) = (\vec{u};\vec{v}) + \pi + 2k\pi$
$(-\vec{u};\vec{v}) = (\vec{u};\vec{v}) + \pi + 2k\pi$

$$(\vec{u};-\vec{v}) - (\vec{u};\vec{v}) = (\vec{u};-\vec{v}) + (\vec{v};\vec{u})$$
$$= (\vec{v};\vec{u}) + (\vec{u};-\vec{v})$$
$$= (\vec{v};-\vec{v}) = \pi + 2k\pi$$





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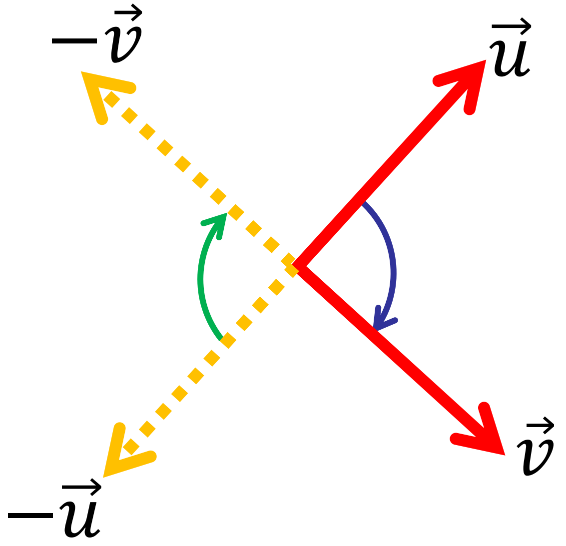
Properties

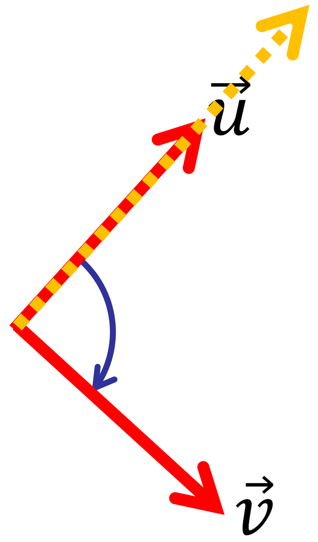
❸ $(-\vec{u};-\vec{v}) = (\vec{u};\vec{v}) + 2k\pi$

$$(-\vec{u};-\vec{v}) = (\vec{u};-\vec{v}) + \pi + 2k\pi$$
$$= (\vec{u};\vec{v}) + \pi + \pi + 2k\pi = (\vec{u};\vec{v}) + 2\pi + 2k\pi$$
$$= (\vec{u};\vec{v}) + 2k'\pi$$



❹ $(a\vec{u};a\vec{v}) = (\vec{u};\vec{v}) + 2k\pi \quad ; \quad a \neq 0$

$(a\vec{u};\vec{v}) = (\vec{u};a\vec{v}) = (\vec{u};\vec{v}) + 2k\pi \quad ; \quad a > 0$

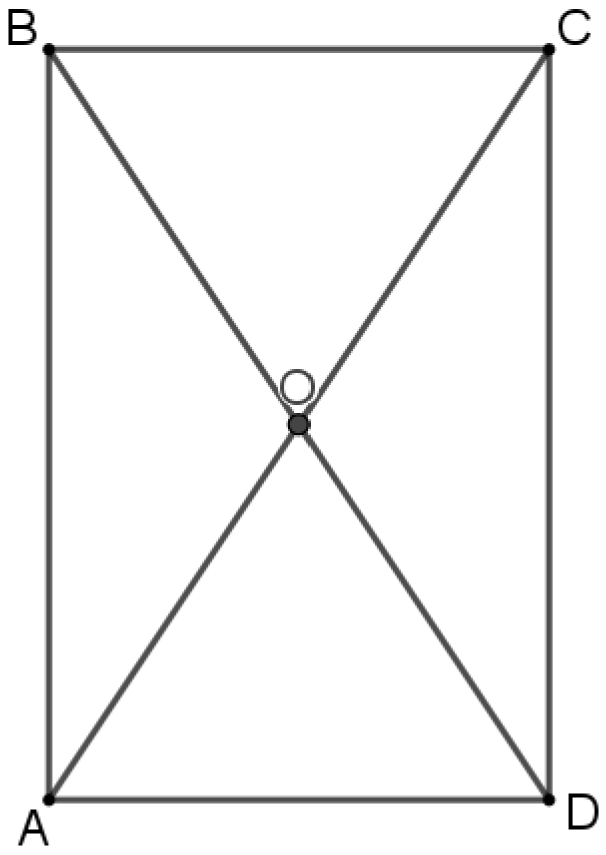


Application # 1

ABCD is a rectangle with center O such that $AB = \sqrt{3}$ and AD=1.
Determine the measure of each of the following directed angles:

$(\overrightarrow{AC}; \overrightarrow{AD}) = -\frac{\pi}{3} \quad (2\pi)$

$\tan \widehat{CAD} = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$



$(\overrightarrow{BA}; \overrightarrow{BD}) =$

$(\overrightarrow{OA}; \overrightarrow{OB}) =$

$(\overrightarrow{OC}; \overrightarrow{AD}) =$

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Application # 1

ABCD is a rectangle with center O such that $AB = \sqrt{3}$ and AD=1.
Determine the measure of each of the following directed angles:

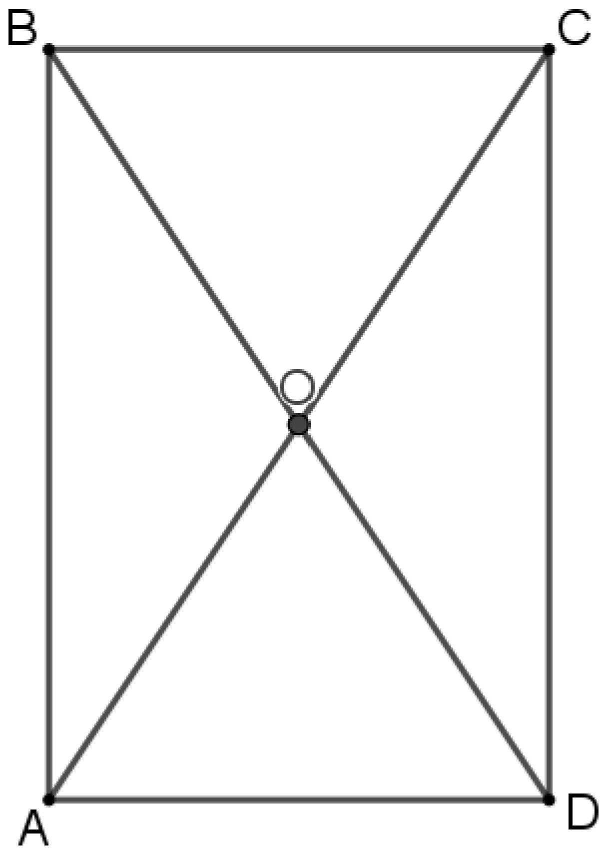
$(\overrightarrow{AC};\overrightarrow{AD}) = -\frac{\pi}{3} \quad (2\pi)$

$(\overrightarrow{BA};\overrightarrow{BD}) = \frac{\pi}{6} \quad (2\pi)$

$\tan \widehat{ABD} = \frac{AD}{AB} = \frac{1}{\sqrt{3}}$

$(\overrightarrow{OA};\overrightarrow{OB}) =$

$(\overrightarrow{OC};\overrightarrow{AD}) =$



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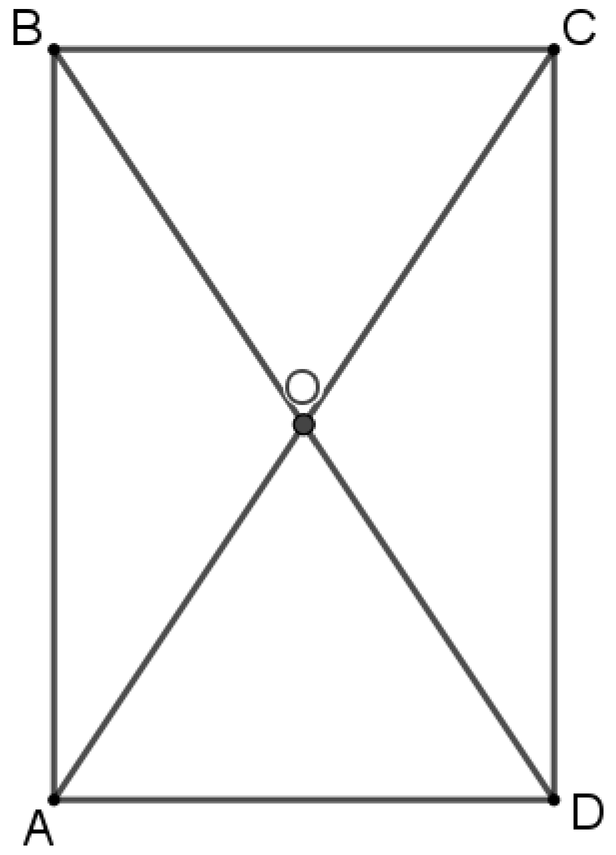
$$(\overrightarrow{AC};\overrightarrow{AD}) = -\frac{\pi}{3} \quad (2\pi)$$

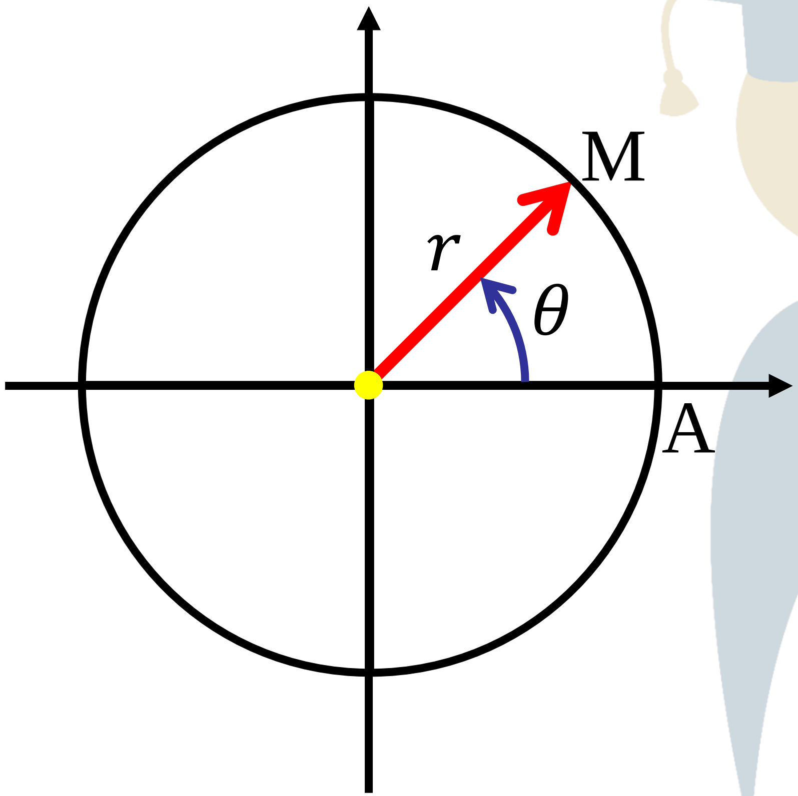
$$(\overrightarrow{BA};\overrightarrow{BD}) = \frac{\pi}{6} \quad (2\pi)$$

$$(\overrightarrow{OA};\overrightarrow{OB}) = -\frac{2\pi}{3} \quad (2\pi)$$

$$(\overrightarrow{OC};\overrightarrow{AD}) = \left(\frac{1}{2}\overrightarrow{AC};\overrightarrow{AD}\right) = (\overrightarrow{AC};\overrightarrow{AD}) = -\frac{\pi}{3} \quad (2\pi)$$



Polar coordinates



r & θ are called the polar coordinates of M.

θ: polar angle

r: radius vector

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

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Application # 2

1) Determine the Cartesian equation of a point of polar coordinates $r = 2 \ \& \ \theta = \frac{\pi}{4}$.

$$x = rcos\theta = 2\cos\frac{\pi}{4} = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = rsin\theta = 2\sin\frac{\pi}{4} = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$$

2) Determine the polar coordinates of a point of Cartesian coordinates $\left(-\frac{1}{2}; -\frac{\sqrt{3}}{2}\right)$.

$$r = \sqrt{x^2 + y^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\tan\theta = \frac{y}{x} = \sqrt{3}$$

$$x < 0 \text{ and } y < 0 \text{ so } \theta = \pi + \frac{\pi}{3} \ \ (2\pi)$$

Solutions of the equation $cos x = cos \alpha$

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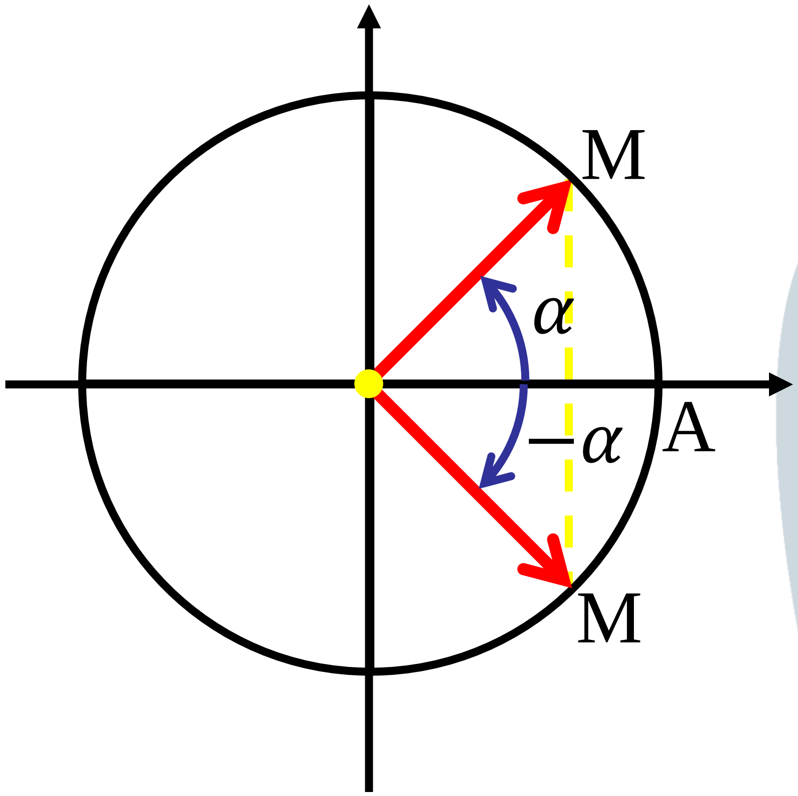
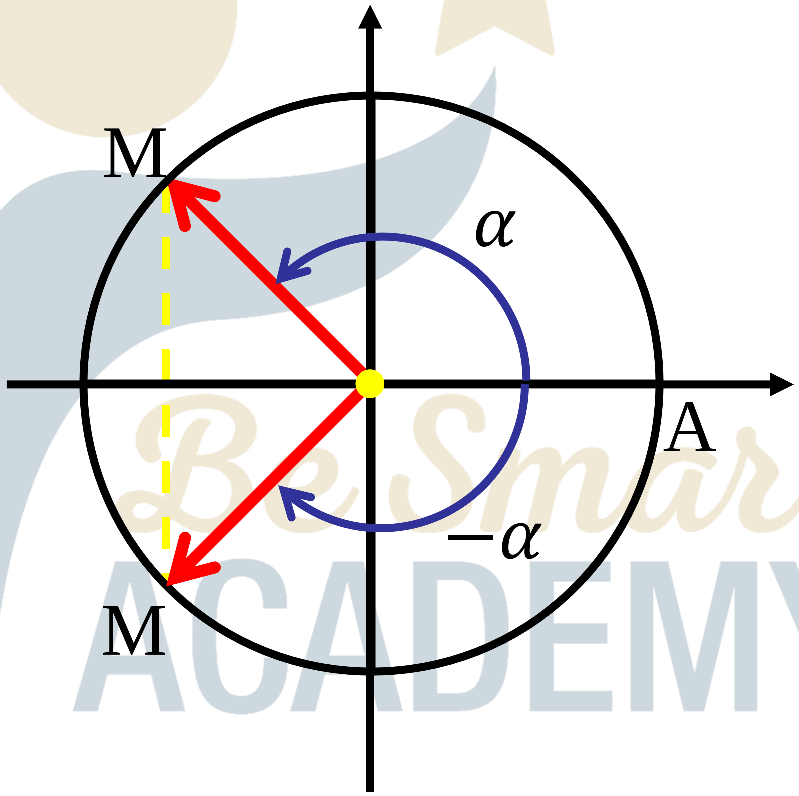
α is an acute angle



α is an obtuse angle



α and $-\alpha$ have same abscissa so,

$\cos \alpha = \cos(-\alpha)$

The equation:

$\cos x = \cos \alpha$ has 2 solutions:

$x = \alpha + 2k\pi$

or

$x = -\alpha + 2k\pi$

Solutions of the equation $\cos x = a$

Case ❶: if $a > 1$ or $a < -1$

The equation has no solution since $-1 \leq \cos x \leq 1$

Example:

$\cos x = 2$; $\cos x = -2$

Case ❷: if $-1 \leq a \leq 1$

The equation has infinity of solutions which have the form:

$x = \alpha + 2k\pi$ or $x = -\alpha + 2k\pi$ where $k \in \mathbb{Z}$ & $\alpha \in [0; \pi]$

Example:

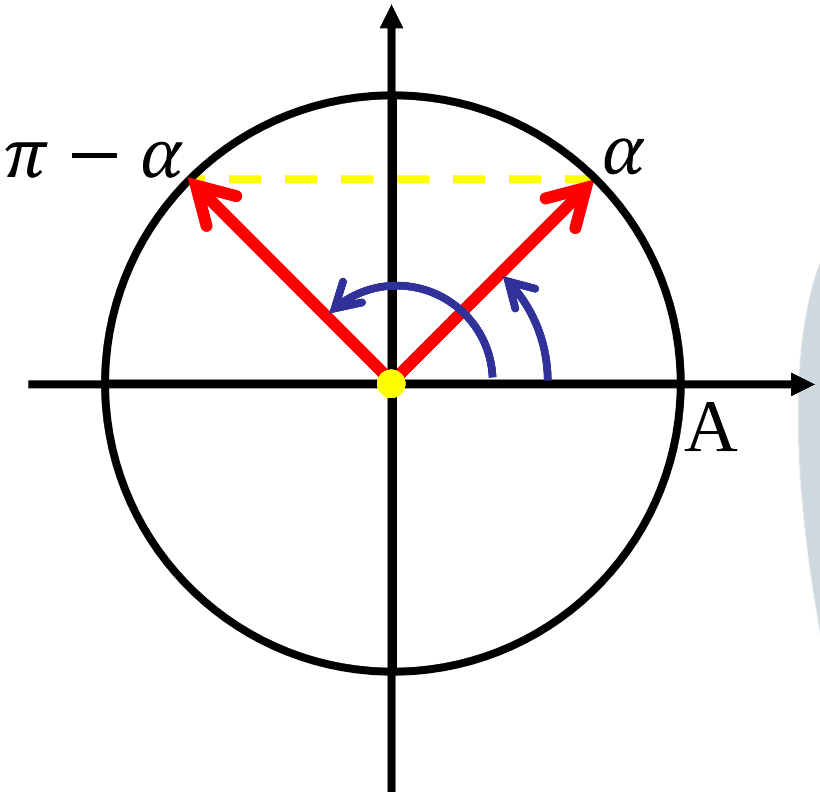
$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

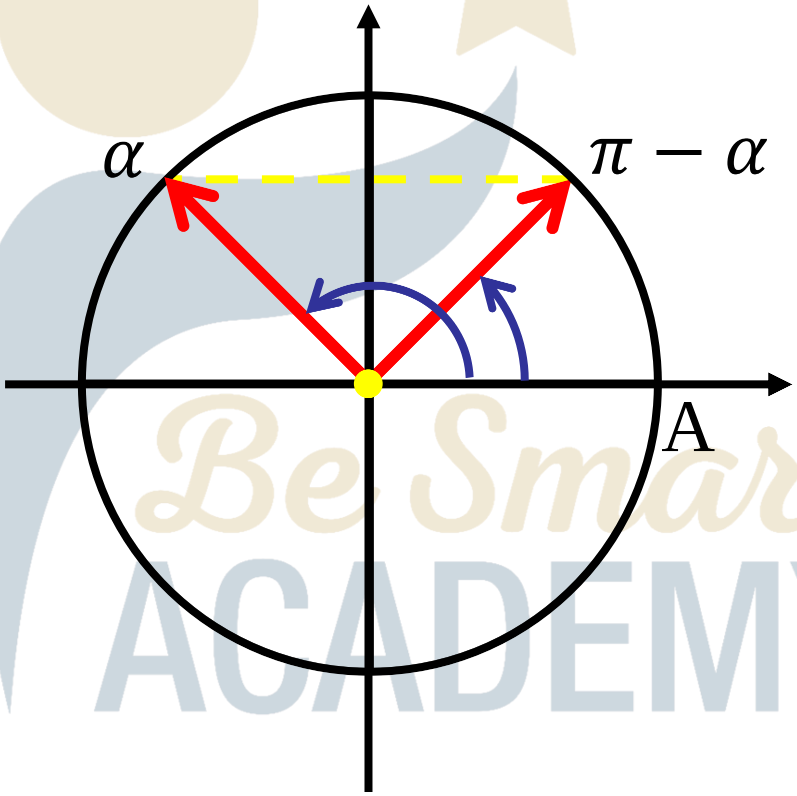
$x = \frac{\pi}{3} + 2k\pi$ or $x = -\frac{\pi}{3} + 2k\pi$

Solutions of the equation $\sin x = \sin\alpha$

α is an acute angle



α is an obtuse angle



α and $\pi - \alpha$ have same ordinates so,

$\sin\alpha = \sin(\pi - \alpha)$

The equation:
$\sin x = \sin\alpha$ has 2 solutions:

$x = \alpha + 2k\pi$

or

$x = \pi - \alpha + 2k\pi$

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Solutions of the equation $\sin x = a$

Case ❶: if $a > 1$ or $a < -1$

The equation has no solution since $-1 \leq \sin x \leq 1$

Example:

$\sin x = 2$; $\sin x = -2$

Case ❷: if $-1 \leq a \leq 1$

The equation has infinity of solutions which have the form:

$x = \alpha + 2k\pi$ or $x = \pi - \alpha + 2k\pi$ where $k \in \mathbb{Z}$ & $\alpha \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

Example:

$\sin x = \frac{1}{2}$

$\sin x = \sin \frac{\pi}{6}$

$x = \frac{\pi}{6} + 2k\pi$ or $x = \pi - \frac{\pi}{6} + 2k\pi = \frac{5\pi}{6} + 2k\pi$

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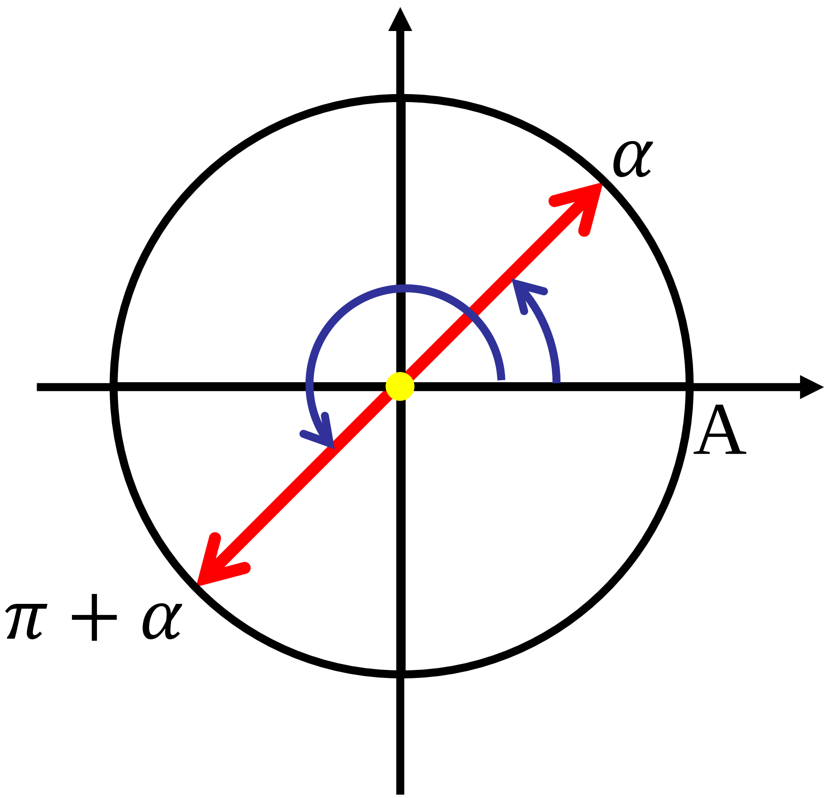
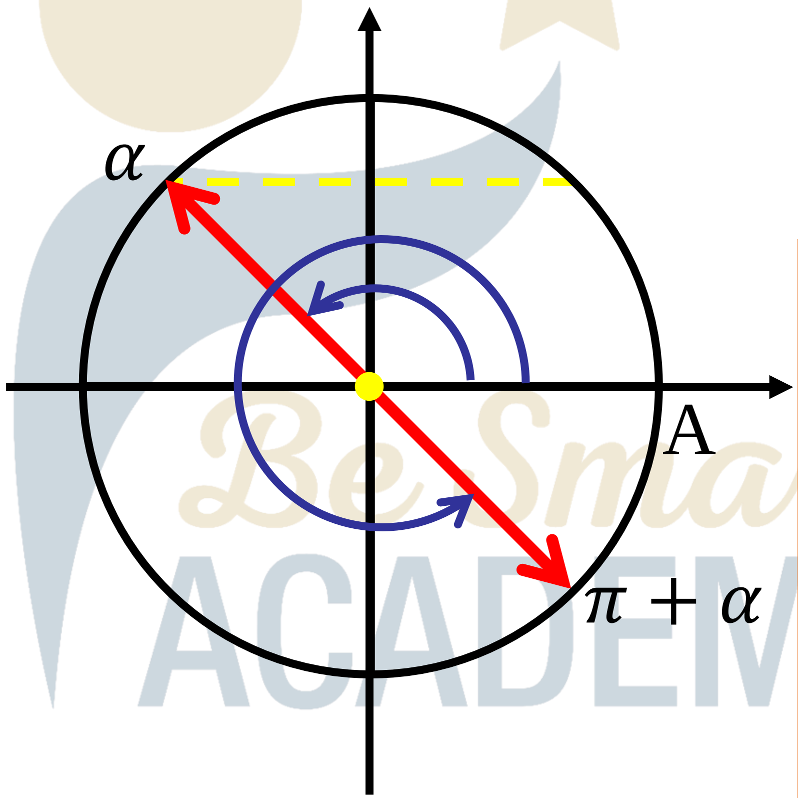
Solutions of the equation $tanx = tan\alpha$

α is an acute angle



α is an obtuse angle



α and $\pi + \alpha$ are in the quadrants where the abscissa and the ordinate have same signs so,

$\tan\alpha = \tan(\pi + \alpha)$

The equation: $\tan x = \tan\alpha$ has 2 solutions:

$x = \alpha + 2k\pi$

or

$x = \pi + \alpha + 2k\pi$

that can be reduced to the form: $x = \alpha + k\pi$

Solutions of the equation $\sin x = a$

The equation has for every real number a the set of solutions: $x = \alpha + k\pi$

Where $k \in \mathbb{Z}$ & $\alpha \in]-\frac{\pi}{2};\frac{\pi}{2}[$

Example:

$\tan x = -\sqrt{3}$

$\tan x = \tan\left(-\frac{\pi}{3}\right)$

$x = -\frac{\pi}{3} + k\pi \quad ; \quad k \in \mathbb{Z}$

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Application # 3

Solve $2\cos^2 x + \sin x - 2 = 0$

$\cos^2 x = 1 - \sin^2 x$

$2(1 - \sin^2 x) + \sin x - 2 = 0$

$2 - 2\sin^2 x + \sin x - 2 = 0$

$-2\sin^2 x + \sin x = 0$

$\sin x\,(-2\sin x + 1) = 0$

$\sin x = 0$ *or* $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2} + 2k\pi$ $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$

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